

# EVENTUAL QUASI-LINEARITY OF THE MINKOWSKI LENGTH

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**ABSTRACT.** The Minkowski length of a lattice polytope  $P$  is a natural generalization of the lattice diameter of  $P$ . It can be defined as the largest number of lattice segments whose Minkowski sum is contained in  $P$ . The famous Ehrhart theorem states that the number of lattice points in the positive integer dilates  $tP$  of a lattice polytope  $P$  behaves polynomially in  $t \in \mathbb{N}$ . In this paper we prove that for any lattice polytope  $P$ , the Minkowski length of  $tP$  for  $t \in \mathbb{N}$  is eventually a quasi-polynomial with linear constituents. We also give a formula for the Minkowski length of coordinate boxes, degree one polytopes, and dilates of unimodular simplices. In addition, we give a new bound for the Minkowski length of lattice polygons and show that the Minkowski length of a lattice triangle coincides with its lattice diameter.

## INTRODUCTION

Let  $P$  be a  $d$ -dimensional lattice polytope in  $\mathbb{R}^d$ . Recall that the lattice diameter  $\ell(P)$  is defined as one less than the largest number of collinear lattice points in  $P$ . The Minkowski length is a natural extension of this notion. For any  $1 \leq n \leq d$ , let  $L_n(P)$  be the largest number of lattice polytopes of positive dimension whose Minkowski sum is at most  $n$ -dimensional and is contained in  $P$ . We call  $L_n(P)$  the  $n$ -th Minkowski length of  $P$ , and  $L(P) = L_d(P)$  simply the Minkowski length of  $P$ . Note  $L_1(P)$  coincides with the lattice diameter  $\ell(P)$ , as in this case the Minkowski summands are collinear lattice segments. It is not hard to show (see the discussion after Definition 1.1) that  $L_n(P)$  is the largest number of lattice segments whose Minkowski sum is at most  $n$ -dimensional and is contained in  $P$ .

The Minkowski length  $L(P)$  of a lattice polytope  $P \subset \mathbb{R}^d$  was first introduced in [6] in relation to studying parameters of toric surface codes. Every lattice polytope  $P$  defines a space  $\mathcal{L}(P)$  of Laurent polynomials (over some field) whose monomials have exponent vectors lying in  $P$ . Such spaces naturally appear in the theory of toric varieties. The algebraic interpretation of the Minkowski length is the following:  $L(P)$  is the largest number of irreducible factors a polynomial  $f \in \mathcal{L}(P)$  may have. This information is particularly important when one studies zeroes of polynomials in  $\mathcal{L}(P)$ , see [4, 5, 6, 7, 8]. A number of results concerning  $L(P)$  appeared in [3, 6, 9].

Let  $tP = \{tx \in \mathbb{R}^d \mid x \in P\}$  be the dilate of  $P$  by a positive integer factor  $t$ . The main result of this paper explains the behavior of  $L_n(tP)$  as a function of the scaling factor  $t \in \mathbb{N}$  in the spirit of the Ehrhart theory. In Theorem 2.20 we prove that for any lattice polytope  $P$  the function  $L_n(tP)$  is eventually quasi-polynomial

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with linear constituents (we say “quasi-linear” for short), which contributes positively to the “ubiquitousness of quasi-polynomials” phenomenon declared by Kevin Woods [10]. For an introduction to the Ehrhart theory we refer the reader to the wonderful book by M. Beck and S. Robins [2].

To prove eventual quasi-linearity of the Minkowski lengths  $L_n(P)$  we define and study their rational counterparts: a sequence of rational numbers  $\lambda_1(P) \leq \dots \leq \lambda_d(P) = \lambda(P)$  associated with  $P$ . Here  $\lambda_1(P)$  is the rational diameter of  $P$  and  $\lambda_n(P)$  is the “asymptotic” Minkowski length, i.e.  $\lambda_n(P) = \lim_{t \rightarrow \infty} L_n(tP)/t$ . In Theorem 2.15 we prove that  $\lambda_n(P) = L_n(kP)/k$  for some  $k \in \mathbb{N}$ .

Although an algorithm for computing  $L(P)$  was presented in dimensions two and three (see [3, 6]), there have been no explicit formulas for  $L(P)$  even for simplices. Here we prove that  $L(t\Delta) = t$  for any unimodular simplex  $\Delta$  and any  $t \in \mathbb{N}$ . This result allowed us to write explicit answers for  $L(P)$  for other classes of polytopes such as coordinate boxes and polytopes of degree one (see Corollary 2.2 and examples afterwards). In Section 3 we prove that for lattice triangles the Minkowski length coincides with the lattice diameter. The final part of the paper contains some examples and open questions.

## 1. PRELIMINARIES

We start with some standard terminology from geometric combinatorics. A polytope  $P \subset \mathbb{R}^d$  is called *lattice* (resp. *rational*) if its vertices have integer (resp. rational) coordinates. A vector  $v \in \mathbb{Z}^d$  is called *primitive* if the greatest common divisor of its coordinates is 1. A lattice segment is called *primitive* if it contains exactly two lattice points. A  $d$ -dimensional simplex is called *unimodular* if its vertices affinely generate the lattice  $\mathbb{Z}^d$ .

Given a lattice polytope  $P \subset \mathbb{R}^d$ , denote by  $\text{Vol}_d(P)$  the Euclidean  $d$ -dimensional volume of  $P$ . Note that the  $d$ -dimensional volume of any parallelepiped formed by a basis of  $\mathbb{Z}^d$  equals 1. More generally, suppose  $P$  is contained in an  $n$ -dimensional rational affine subspace  $a + H$  for a rational linear subspace  $H \subset \mathbb{R}^d$  and  $a \in \mathbb{Q}^d$ . Denote by  $\text{Vol}_n(P)$  the  $n$ -dimensional volume of  $P$  normalized such that the  $n$ -dimensional volume of any parallelepiped formed by a basis of the lattice  $H \cap \mathbb{Z}^d$  equals 1.

**1.1. Minkowski length.** Let  $P$  and  $Q$  be convex polytopes in  $\mathbb{R}^d$ . Their *Minkowski sum* is the set

$$P + Q = \{p + q \in \mathbb{R}^d \mid p \in P, q \in Q\},$$

which is again a convex polytope.

**Definition 1.1.** Let  $P$  be a lattice polytope in  $\mathbb{R}^d$ . Define the *Minkowski length*  $L = L(P)$  of  $P$  to be the largest number of lattice polytopes  $Q_1, \dots, Q_L$  of positive dimension whose Minkowski sum is contained in  $P$ . Any such sum  $Q_1 + \dots + Q_L$  is called a *maximal decomposition* in  $P$ .

We refer the reader to [3] for examples illustrating this definition. It is clear from the definition that  $L(P)$  is monotone with respect to inclusion:  $L(P) \leq L(Q)$  if  $P \subseteq Q$ , and is superadditive with respect to the Minkowski sum:  $L(P + Q) \geq L(P) +$

$L(Q)$ . Also,  $L(P)$  is invariant under unimodular transformations (isomorphisms of the lattice  $\mathbb{Z}^d$ ).

There is a natural partial order on the set of maximal decompositions in  $P$ , as defined in [3]. Namely, we say that

$$Q_1 + \cdots + Q_L \leq P_1 + \cdots + P_L$$

if  $Q_1 + \cdots + Q_L$  is contained in  $P_1 + \cdots + P_L$  after a possible lattice translation. Minimal elements with respect to this partial order are called *smallest maximal decompositions*. Clearly, every smallest maximal decomposition is the Minkowski sum of  $L$  lattice segments, i.e. is a lattice zonotope.

Any lattice (resp. rational) zonotope  $Z$  can be written in the form

$$(1.1) \quad Z = a + \alpha_1[0, v_1] + \cdots + \alpha_m[0, v_m],$$

for some  $m \in \mathbb{N}$ , distinct primitive vectors  $v_i \in \mathbb{Z}^d$ , positive integer (resp. rational) numbers  $\alpha_i$ , and a lattice (resp. rational) point  $a \in \mathbb{R}^d$ . In this case, we set

$$|Z| := \alpha_1 + \cdots + \alpha_m.$$

The following result from [3] gives a universal bound for the number of distinct summands in a smallest maximal decomposition.

**Proposition 1.2.** [3] *Let  $P \subset \mathbb{R}^d$  be a lattice polytope. Then every smallest maximal decomposition in  $P$  has at most  $2^d - 1$  distinct summands.*

**1.2. Quasi-polynomials.** Here we recall the definition of a quasi-polynomial function.

**Definition 1.3.** A function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  is called a *quasi-polynomial* if there exist  $k \in \mathbb{N}$  and polynomials  $p_0, \dots, p_{k-1} \in \mathbb{Q}[t]$ , called the *constituents* of  $f$ , such that

$$f(t) = p_r(t) \quad \text{whenever } t \equiv r \pmod{k}, \text{ for } 0 \leq r \leq k.$$

The smallest such  $k$  is called the *period* of  $f$ . If all the constituents of  $f$  are linear we say that  $f$  is *quasi-linear*. Finally, we say  $f : \mathbb{N} \rightarrow \mathbb{Q}$  is *eventually quasi-linear* if  $f(t)$  coincides with a quasi-linear function for all large enough  $t$ .

**Example 1.4.** The function  $f(t) = 3 \lfloor \frac{t}{3} \rfloor + 4$  is quasi-linear with period  $k = 3$ , where  $\lfloor x \rfloor$  denotes the floor of  $x$ . Indeed,

$$f(t) = \begin{cases} t + 4, & \text{if } t \equiv 0 \pmod{3} \\ t + 3, & \text{if } t \equiv 1 \pmod{3} \\ t + 2, & \text{if } t \equiv 2 \pmod{3}. \end{cases}$$

## 2. MAIN THEOREMS

Before we prove our main result about eventual quasi-linearity of the Minkowski length, we will look at some instances when it is, in fact, linear. The simplest such example is when  $P = \Delta$ , a unimodular  $d$ -simplex.

**Theorem 2.1.** *Let  $\Delta$  be a unimodular  $d$ -simplex and  $t \in \mathbb{N}$ . Then*

$$L(t\Delta) = tL(\Delta) = t.$$

*Proof.* After a unimodular transformation we may assume that  $\Delta$  is the standard  $d$ -simplex, i.e. the convex hull of  $\{0, e_1, \dots, e_d\}$  where  $e_i$  are the standard basis vectors. First, it is easy to see that  $L(\Delta) = 1$ . Also,  $L(t\Delta) \geq t$  as  $t\Delta$  contains the Minkowski sum of  $t$  lattice segments  $[0, e_1]$ .

We prove the converse by induction on  $d$ . The case  $d = 1$  is trivial. Denote  $L = L(t\Delta)$ . Let  $Z$  be a smallest maximal decomposition in  $t\Delta$  and let  $a \in Z$  be a vertex with the smallest sum of the coordinates, which we denote by  $\alpha$ . We have

$$Z = a + [0, v_1] + \dots + [0, v_L],$$

where  $v_i \in \mathbb{Z}^d$  are primitive, not necessarily distinct vectors. Note that the sum of the coordinates of each  $v_i$  is non-negative, by the choice of  $a$ . Suppose the first  $k$  of the vectors  $v_1, \dots, v_L$  have the sum of the coordinates equal zero, for  $0 \leq k \leq L$ . Then the subzonotope

$$Z' = a + [0, v_1] + \dots + [0, v_k]$$

is contained in  $\alpha\Delta'$ , where  $\Delta'$  is the facet of  $\Delta$  whose points have the sum of the coordinates equal to 1. This implies  $k \leq L(\alpha\Delta')$ . By induction  $L(\alpha\Delta') = \alpha$ , hence,  $k \leq \alpha$ . Now the point  $v = a + v_{k+1} + \dots + v_L$  lies in  $Z$  and has the sum of the coordinates at least  $\alpha + L - k \geq L$ . On the other hand,  $v$  lies in  $t\Delta$ , so its sum of the coordinates is at most  $t$ . Therefore,  $L \leq t$ .  $\square$

**Corollary 2.2.** *Let  $P$  be a lattice polytope contained in  $\alpha\Delta$  for some unimodular simplex  $\Delta$  and  $\alpha \in \mathbb{N}$ . If  $P$  contains the Minkowski sum of  $\alpha$  lattice segments then  $L(P) = \alpha$ . Consequently,  $L(tP) = tL(P)$ .*

**Example 2.3.** Let  $\Pi$  be a lattice coordinate box in  $\mathbb{R}^d$ , i.e.  $\Pi = [0, \alpha_1 e_1] \times \dots \times [0, \alpha_d e_d]$ , where  $e_i$  are the standard basis vectors and  $\alpha_i$  are non-negative integers. Clearly,  $\Pi = \alpha_1[0, e_1] + \dots + \alpha_d[0, e_d]$  and  $\Pi$  is contained in  $(\alpha_1 + \dots + \alpha_d)\Delta_d$ , where  $\Delta_d$  is the standard  $d$ -simplex. Therefore,

$$L(\Pi) = \alpha_1 + \dots + \alpha_d.$$

We also have  $L(t\Pi) = tL(\Pi)$ .

**Example 2.4.** According to a result of Batyrev and Nill [1], a  $d$ -dimensional polytope  $P$  has *degree one* if and only if  $P$  is either

- (1) the  $d - 2$  iterated pyramid over the triangle  $\Delta_2 = \text{Conv}\{(0, 0), (2, 0), (0, 2)\}$ ,  
or
- (2) the  $d - n$  iterated pyramid over a Lawrence prism  $Q$  defined by a sequence of integers  $0 < h_1 \leq \dots \leq h_n$ :

$$(2.1) \quad Q = \text{Conv}\{0, e_1, \dots, e_{n-1}, e_1 + h_1 e_n, \dots, e_{n-1} + h_{n-1} e_n, h_n e_n\} \subset \mathbb{R}^n.$$

Corollary 2.2 implies that in the first case  $L(P) = 2$  since  $P$  contains the segment  $[0, 2e_1]$  and  $P \subset 2\Delta_d$ . In the second case

$$L(P) = \begin{cases} h_n, & \text{if } h_{n-1} < h_n \\ h_n + 1, & \text{if } h_{n-1} = h_n. \end{cases}$$

Indeed, if  $h_{n-1} < h_n$  then  $P \subset h_n \Delta_d$  and  $P$  contains the segment  $[0, h_n e_n]$ . If  $h_{n-1} = h_n$  then  $P \subset (h_n + 1)\Delta_d$  and  $P$  contains the rectangle  $[0, e_{n-1}] + [0, h_n e_n]$ .

**2.1. Rational Minkowski length.** Let  $P$  be an arbitrary lattice polytope in  $\mathbb{R}^d$ . The following is a generalization of Definition 1.1.

**Definition 2.5.** Let  $P$  be a lattice polytope in  $\mathbb{R}^d$ . Define the  $n$ -th Minkowski length  $L = L_n(P)$  of  $P$  to be the largest number of lattice polytopes  $Q_1, \dots, Q_L$  of positive dimension whose Minkowski sum is at most  $n$ -dimensional and is contained in  $P$ .

Clearly,  $L_1(P) \leq \dots \leq L_{d-1}(P) \leq L_d(P) = L(P)$ . Note  $L_1(P)$  coincides with the lattice diameter  $\ell(P)$ , which is defined as one less than the largest number of collinear lattice points in  $P$ .

**Example 2.6.** Let  $\square$  be the unit square in  $\mathbb{R}^2$ . Then

$$L_1(\square) = 1 \quad \text{and} \quad L_2(\square) = L(\square) = 2.$$

For any unimodular  $d$ -simplex  $\Delta$  and any  $t \in \mathbb{N}$  we have  $L_1(t\Delta) \geq t$  as  $t\Delta$  contains the segment  $[0, te_1]$ . By Theorem 2.1,  $L(t\Delta) = t$ , hence

$$L_1(t\Delta) = \dots = L_d(t\Delta) = L(t\Delta) = t.$$

If  $P$  is the  $d-n$  iterated pyramid over a Lawrence prism  $Q$  as in (2.1) then  $L_1(P) = h_n$  and  $L_2(P) = \dots = L_d(P) = L(P)$ .

The following is a rational analog of the Minkowski length.

**Definition 2.7.** The number

$$\lambda(P) = \sup_{t \in \mathbb{N}} \frac{L(tP)}{t}$$

is called the *rational Minkowski length* of  $P$ . More generally, the  $n$ -th rational Minkowski length of  $P$  is

$$\lambda_n(P) = \sup_{t \in \mathbb{N}} \frac{L_n(tP)}{t}$$

The following proposition asserts that the numbers  $\lambda_n(P)$  are well-defined.

**Proposition 2.8.** For any  $t \in \mathbb{N}$  we have  $L_n(tP) \leq t\alpha$ , where  $\alpha \in \mathbb{N}$  is such that  $P \subseteq \alpha\Delta$  for a unimodular simplex  $\Delta$ .

*Proof.* It is enough to consider the case  $n = d$ . Then it is immediate from Theorem 2.1:  $L(tP) \leq L(t\alpha\Delta) = t\alpha$ .  $\square$

It follows from the definition that  $\lambda_1(P) \leq \dots \leq \lambda_{d-1}(P) \leq \lambda_d(P) = \lambda(P)$ .

**Remark 2.9.** As  $L_n(tP)$  satisfies the superadditivity property, the supremum in the above definition may be replaced with the limit, by Fekete's lemma. We will not be using this result in our further discussion.

**Definition 2.10.** Let  $K \subset \mathbb{R}^d$  be a rational polytope. For any primitive vector  $v \in \mathbb{Z}^d$  define  $s_v(K)$  to be the largest rational number  $s$  such that the segment  $[0, sv]$  is contained in  $K$  after a translation by a rational vector. The *rational diameter*  $s(K)$  is the maximum of  $s_v(K)$  over all primitive  $v \in \mathbb{Z}^d$ .

It is not hard to see that  $s(P) = \lambda_1(P)$  for any lattice polytope  $P$ . Indeed, for any  $t \in \mathbb{N}$ , the polytope  $P$  contains a segment  $a + [0, sv]$  for some  $a \in \mathbb{Q}^d$ , primitive  $v \in \mathbb{Z}^d$ , and  $s = L_1(tP)/t$ . Thus,  $\lambda_1(P) \leq s(P)$ . Conversely, if  $a + [0, s(P)v]$  is contained in  $P$  for some  $a \in \mathbb{Q}^d$  and primitive  $v \in \mathbb{Z}^d$  then there exists  $t \in \mathbb{N}$  such that  $ta + [0, ts(P)v]$  is a lattice segment contained in  $tP$ , i.e.  $s(P) \leq L_1(tP)/t \leq \lambda_1(P)$ . As a corollary, we obtain  $L_1(P) = \lfloor \lambda_1(P) \rfloor$ , as  $\ell(P) = \lfloor s(P) \rfloor$ .

In our main theorem below (Theorem 2.15) we show that  $\lambda(P)$  as well as all  $\lambda_n(P)$  are, in fact, rational numbers. First, we need a few lemmas.

**Lemma 2.11.** *Let  $K$  be a convex body in  $\mathbb{R}^d$  and fix  $\varepsilon > 0$ . Then the set*

$$U_\varepsilon(K) = \{v \in \mathbb{Z}^d \mid v \text{ primitive, } s_v(K) \geq \varepsilon\}$$

*is finite.*

*Proof.* First, note that if  $K \subseteq K'$  then  $s_v(K) \leq s_v(K')$ , and  $s_v(\alpha K) = \alpha s_v(K)$  for  $\alpha \in \mathbb{Q}$ . Thus it is enough to prove the statement for  $K = \mathbb{B}$ , the  $d$ -dimensional unit ball. Let  $v \in \mathbb{Z}^d$  be primitive. By definition  $s_v(\mathbb{B})$  is the number  $s \in \mathbb{Q}$  such that  $\|sv\| = 2$ , where  $\|\cdot\|$  is the usual Euclidean norm. It follows that  $s_v(\mathbb{B}) \geq \varepsilon$  if and only if  $\|v\| \leq 2/\varepsilon$ . In other words,  $U_\varepsilon(\mathbb{B})$  is a lattice set contained in the ball of radius  $2/\varepsilon$  and so is finite.  $\square$

**Lemma 2.12.** *Let  $Z = a + \alpha_1[0, v_1] + \dots + \alpha_m[0, v_m]$  be a smallest maximal decomposition in  $P$  and  $n = \dim Z$ . Then for any  $1 \leq i_1 < \dots < i_n \leq m$  the  $n$ -dimensional volume of the parallelepiped formed by  $v_{i_1}, \dots, v_{i_n}$  is no greater than  $n^d$ .*

*Proof.* We may assume that  $v_{i_1}, \dots, v_{i_n}$  are linearly independent. Let  $a + H$  be the affine span of  $Z$  and let  $\mathbb{L} = H \cap \mathbb{Z}^d$  be the corresponding lattice of rank  $n$ . It is well-known that the  $n$ -dimensional volume of the parallelepiped formed by  $n$  linearly independent lattice vectors  $w_1, \dots, w_n \in \mathbb{L}$  equals the number of lattice points in the half-open parallelepiped

$$\{\lambda_1 w_1 + \dots + \lambda_n w_n \mid 0 \leq \lambda_i < 1 \text{ for } 1 \leq i \leq n\},$$

which is less than the number of lattice points in the closed parallelepiped.

Let  $\{v_{i_1}, \dots, v_{i_n}\}$  be a subset of the set of vectors appearing in the decomposition  $Z$ . We claim that the parallelepiped  $\Pi$  they form has at most  $n^d$  lattice points. Indeed, consider the image of  $\Pi \cap \mathbb{Z}^d$  in  $(\mathbb{Z}/n\mathbb{Z})^d$ . If there are two lattice points in  $\Pi$  congruent mod  $(n\mathbb{Z})^d$  then the lattice segment  $E$  containing them lies in  $\Pi$  and has lattice length  $n$ . Therefore, if we replace  $[0, v_{i_1}] + \dots + [0, v_{i_n}]$  in the decomposition  $Z$  with  $E$  we obtain a maximal decomposition  $Z'$  properly contained in  $Z$ . This contradicts the fact that  $Z$  is a smallest maximal decomposition. This shows that all lattice points of  $\Pi$  are different in  $(\mathbb{Z}/n\mathbb{Z})^d$ , i.e. their number cannot exceed  $n^d$ .  $\square$

**Lemma 2.13.** *Let  $B = \{u_1, \dots, u_n\}$  be a basis for a rational linear subspace  $H \subseteq \mathbb{R}^d$ , and fix a constant  $N$ . Let  $\text{Vol}_n(u_1, \dots, v_i, \dots, u_n)$  denote the  $n$ -dimensional volume of the parallelepiped formed by  $u_1, \dots, u_n$  where  $u_i$  is replaced with a vector  $v$ . Then the set*

$$V(B) = \{v \in H \cap \mathbb{Z}^d \mid \text{Vol}_n(u_1, \dots, v_i, \dots, u_n) \leq N, \text{ for all } 1 \leq i \leq n\}$$

*is finite.*

*Proof.* Fix a coordinate system in  $H$  by choosing a basis for the lattice  $H \cap \mathbb{Z}^d$ . Write  $v = x_1 u_1 + \cdots + x_n u_n$  for some  $x_i \in \mathbb{R}$ . Then by Cramer's rule

$$|x_i| = \frac{\text{Vol}_n(u_1, \dots, v, \dots, u_n)}{\text{Vol}_n(u_1, \dots, u_n)} \leq \frac{N}{\text{Vol}_n(u_1, \dots, u_n)} =: c_i.$$

Therefore, the set  $V(B)$  is contained in the set of lattice points of the parallelepiped formed by  $\{\pm c_1 u_1, \dots, \pm c_n u_n\}$ , and, hence, is finite.  $\square$

**Lemma 2.14.** *Let  $P \subset \mathbb{R}^d$  be a lattice polytope. Fix an ordered collection of primitive vectors  $\mathbf{v} = (v_1, \dots, v_m) \in (\mathbb{Z}^d)^m$ . Then the set of zonotopes*

$$\mathcal{Z}(\mathbf{v}) = \{Z = a + \alpha_1[0, v_1] + \cdots + \alpha_m[0, v_m] \mid a \in \mathbb{R}^d, \alpha_i \in \mathbb{R}_{\geq 0}, Z \subseteq P\}$$

*is a rational polytope in  $\mathbb{R}^{d+m}$ . The function  $|\cdot| : \mathcal{Z}(\mathbf{v}) \rightarrow \mathbb{R}$ ,  $Z \mapsto |Z|$  is an integer linear function on  $\mathcal{Z}(\mathbf{v})$ .*

*Proof.* With every such zonotope  $Z$  we associate a point  $z = (a, \alpha_1, \dots, \alpha_m) \in \mathbb{R}^{d+m}$ . Note that  $Z$  is the convex hull of the following set of  $2^m$  points in  $\mathbb{R}^d$ :

$$K = \left\{ a + \sum_{i \in I} \alpha_i v_i \mid I \subseteq \{1, \dots, m\} \right\}.$$

Clearly  $Z \subseteq P$  if and only if  $K \subset P$  which is expressed by  $2^m$  rational linear inequalities in  $d + m$  variables. Therefore, they define a rational polytope  $\mathcal{Z}(\mathbf{v})$  in  $\mathbb{R}^{d+m}$  (the boundedness of  $\mathcal{Z}(\mathbf{v})$  follows from that of  $P$ ).

The function  $|\cdot| : \mathcal{Z}(\mathbf{v}) \rightarrow \mathbb{R}$ ,  $Z \mapsto |Z|$  is determined by the sum of the last  $m$  coordinates in  $\mathbb{R}^{d+m}$ , hence, is an integer linear function on  $\mathcal{Z}(\mathbf{v})$ .  $\square$

Notice that reordering of the  $v_i$  does not change the zonotope  $Z$ , so the polytope  $\mathcal{Z}(\mathbf{v})$ , as well as the function  $|\cdot| : \mathcal{Z}(\mathbf{v}) \rightarrow \mathbb{R}$ , is invariant under permutations of the last  $m$  coordinates.

Now we are ready for our main result.

**Theorem 2.15.** *Let  $P \subset \mathbb{R}^d$  be a lattice polytope. Then*

$$\lambda(P) = \frac{L(kP)}{k},$$

*for some  $k \in \mathbb{N}$ .*

*Proof.* Consider the polytope  $tP$  for some  $t \in \mathbb{N}$ . It follows from Proposition 1.2, that  $tP$  contains a smallest maximal decomposition  $Z$  with  $m \leq M := 2^d - 1$  distinct summands

$$(2.2) \quad Z = a + \alpha_1[0, v_1] + \cdots + \alpha_m[0, v_m],$$

where  $a \in \mathbb{Z}^d$ ,  $v_i \in \mathbb{Z}^d$  are primitive, and  $\alpha_i$  are positive integers whose sum equals the Minkowski length  $L(tP)$ . Therefore,  $P$  contains a rational zonotope

$$Z/t = a/t + (\alpha_1/t)[0, v_1] + \cdots + (\alpha_m/t)[0, v_m]$$

with  $|Z/t| = L(tP)/t$ . Conversely, every rational zonotope  $Z$  in  $P$  has the form

$$(2.3) \quad Z = a + \alpha_1[0, v_1] + \cdots + \alpha_m[0, v_m],$$

for some  $a \in \mathbb{Q}^d$ , primitive  $v_i \in \mathbb{Z}^d$ , and non-negative rationals  $\alpha_i$ . Then there exists  $t \in \mathbb{N}$  such that  $tZ$  is a lattice zonotope in  $tP$ , and so  $|Z| \leq L(tP)/t \leq \lambda(P)$ . Therefore,  $\lambda(P)$  is the supremum of the function  $Z \mapsto |Z|$  on the set of all rational zonotopes  $Z$  contained in  $P$ .

We will show below that there exist  $\delta > 0$ , independent of  $t$ , and a finite set of primitive vectors  $V_\delta \subset \mathbb{Z}^d$  satisfying the following property: If  $Z$  is a smallest maximal decomposition in  $tP$  for some  $t \in \mathbb{N}$  then

$$\lambda(P) - \delta < |Z/t| \leq \lambda(P)$$

implies that  $v_1, \dots, v_m$  lie in  $V_\delta$ . By Lemma 2.14,  $\lambda(P)$  equals the maximum of the linear function  $Z \mapsto |Z|$  on the union of rational polytopes  $\mathcal{Z}(\mathbf{v})$  over all collections  $\mathbf{v} = (v_1, \dots, v_m) \in (V_\delta)^m$ , hence,  $\lambda(P) = |Z'|$  for some rational zonotope  $Z' \subset P$ . Choose  $k \in \mathbb{N}$  such that  $kZ'$  is a lattice zonotope in  $kP$ . Then

$$\lambda(P) = \frac{L(kP)}{k},$$

as required.

It remains to prove the existence of  $\delta > 0$  and  $V_\delta$  satisfying the above property. Denote  $\lambda = \lambda(P)$ , and  $\lambda_n = \lambda_n(P)$ , the  $n$ -th rational Minkowski length of  $P$ . Let  $e \geq 1$  be the smallest integer such that  $\lambda_n = \lambda(P)$  for all  $n \geq e$ . Then to find  $\lambda$  it is enough to consider only smallest maximal decompositions in  $tP$  of dimension at most  $e$ . Let  $Z$  be such a decomposition, as in (2.2).

The case  $e = 1$  is easy — we set  $\delta = \lambda/2$  and  $V_\delta = U_\delta(P)$ , as in Lemma 2.11, which is a finite set. In this case  $|Z/t| = \alpha_1/t \leq s_{v_1}(P) < \delta = \lambda/2$ , unless  $v_1 \in V_\delta$ .

If  $e > 1$  we have

$$\lambda_1 \leq \dots \leq \lambda_{e-1} < \lambda_e = \dots = \lambda_d = \lambda.$$

Set  $\delta = (\lambda - \lambda_{e-1})/2$  and choose

$$0 < \varepsilon < \min \left\{ \frac{\lambda - \delta}{M}, \frac{\delta}{M - e} \right\}.$$

If no  $v_i$  lies in  $U_\varepsilon(P)$  then  $\alpha_i/t \leq s_{v_i}(P) < \varepsilon$ , and so  $|Z/t| \leq m\varepsilon < \lambda - \delta$ . Thus, we may assume that  $v_i \in U_\varepsilon(P)$  for  $1 \leq i \leq k$  and  $v_i \notin U_\varepsilon(P)$  for  $k < i \leq m$ .

First, suppose that  $\{v_1, \dots, v_k\}$  spans an  $e$ -dimensional subspace. By Lemmas 2.12 and 2.13, there are only finitely many choices for each  $v_i$  for  $k < i \leq m$ . Thus we define  $V_\delta$  to be the union of  $U_\varepsilon(P)$  and the finite sets  $V(B)$  for every subset  $B = \{u_1, \dots, u_e\} \subset U_\varepsilon(P)$  which spans an  $e$ -dimensional subspace.

Next, suppose the dimension of the span of  $\{v_1, \dots, v_k\}$  is less than  $e$ . We may assume that  $v_l, \dots, v_m$  lie outside of this span for some  $k < l \leq m$ . Then we have

$$|Z/t| \leq \lambda_{e-1} + (\alpha_l/t) + \dots + (\alpha_m/t) < \lambda_{e-1} + (m - l)\varepsilon.$$

By the choice of  $\varepsilon$ , and since  $l > e$ , the latter is smaller than  $\lambda - \delta$ .  $\square$

**Remark 2.16.** The same arguments as above show that for any  $1 \leq n \leq d$ ,

$$\lambda_n(P) = \frac{L(k_n P)}{k_n},$$

for some  $k_n \in \mathbb{N}$ . In particular, all  $\lambda_n(P)$  are rational numbers.



**2.2. Quasi-linearity of the Minkowski length.** The result of Theorem 2.15 allows us to make the following definition.

**Definition 2.17.** The smallest  $k \in \mathbb{N}$  satisfying  $\lambda(P) = L(kP)/k$  is called the *period* of  $P$ .

In Theorem 2.20 we prove that the Minkowski length is eventually quasi-linear, but first we are going to show that the rational Minkowski length is linear (Proposition 2.19). We will need the following lemma.

**Lemma 2.18.** *Let  $k$  be the period of  $P$ . Then  $L(tP) = t\lambda(P)$  whenever  $k \mid t$ ,  $t \in \mathbb{N}$ .*

*Proof.* Since  $tP$  contains the Minkowski sum of  $t/k$  copies of  $kP$ , we have

$$L(tP) \geq (t/k)L(kP) = t\lambda(P).$$

On the other hand,  $t\lambda(P) \geq L(tP)$ , by the definition of  $\lambda(P)$ . □

**Proposition 2.19.** *For any  $t \in \mathbb{N}$  we have  $\lambda(tP) = t\lambda(P)$ .*

*Proof.* We have

$$\lambda(tP) = \max_{s \in \mathbb{N}} \frac{L(stP)}{s} = t \max_{s \in \mathbb{N}} \frac{L(stP)}{st} \leq t\lambda(P).$$

On the other hand, by Theorem 2.15 and Lemma 2.18,

$$t\lambda(P) = \frac{kt\lambda(P)}{k} = \frac{L(ktP)}{k} \leq \lambda(tP),$$

where  $k$  is the period of  $P$ . □

**Theorem 2.20.** *Let  $P$  be a lattice polytope in  $\mathbb{R}^d$  with period  $k$ . Then the function  $L(tP)$  is eventually quasi-linear with period at most  $k$ . More explicitly, there exist integers  $c_r$ , for  $0 \leq r < k$ , such that  $L(rP) \leq c_r \leq r\lambda(P)$  and*

$$L(tP) = k\lambda(P) \left\lfloor \frac{t}{k} \right\rfloor + c_r$$

*whenever  $t \equiv r \pmod{k}$  and  $t$  is large enough.*

*Proof.* Fix  $0 \leq r < k$  and let  $t \equiv r \pmod{k}$ . Denote  $c_r(t) = L(tP) - k\lambda(P) \left\lfloor \frac{t}{k} \right\rfloor$ . Note that  $c_r(t)$  is an integer as  $k\lambda(P) = L(kP)$ . We will show that  $c_r(t)$  is constant for  $t$  large enough. Indeed,  $c_r(t)$  are bounded from above:

$$c_r(t) = L(tP) - k\lambda(P) \left\lfloor \frac{t}{k} \right\rfloor \leq t\lambda(P) - k\lambda(P) \left\lfloor \frac{t}{k} \right\rfloor = r\lambda(P).$$

Also they increase:

$$c_r(t+k) = L(tP + kP) - k\lambda(P) \left\lfloor \frac{t+k}{k} \right\rfloor \geq L(tP) + L(kP) - k\lambda(P) \left\lfloor \frac{t+k}{k} \right\rfloor = c_r(t),$$

where we used  $L(kP) = k\lambda(P)$  in the last equality. Therefore, the integers  $c_r(t)$  for  $t \equiv r \pmod{k}$  eventually stabilize to a constant  $c_r$ .

We have already seen that  $c_r \leq r\lambda(P)$ . For the other inequality, let  $t = qk + r$ . Then, using Lemma 2.18, we obtain

$$c_r(t) = L(tP) - qk\lambda(P) \geq L(qkP) + L(rP) - qk\lambda(P) = L(rP).$$

□

**Remark 2.21.** The above proof works just as well if we replace  $L(P)$  with  $L_n(P)$  for any  $1 \leq n \leq d$  and apply Remark 2.16. Therefore, each  $n$ -th Minkowski length of  $P$  is eventually quasi-linear with period at most  $k_n$ . Since  $L_1(P) = \lfloor \lambda_1(P) \rfloor$ , the function  $L_1(tP)$  is, in fact, quasi-linear.

### 3. DIMENSION TWO

In this section we deal with lattice polytopes in dimension two. We prove an upper bound on the rational length of  $P$  in terms of other well-known invariants of  $P$  — the Euclidean area and the lattice width of  $P$ . As an application we give a formula for  $\lambda(tP)$  and  $L(tP)$  for any triangle  $P$  in  $\mathbb{R}^2$ .

Let  $P \subset \mathbb{R}^d$  be a lattice polytope and  $v \in \mathbb{Z}^d$  a primitive vector. Recall that the *lattice width of  $P$  in the direction  $v$*  is the integer  $\max_{x \in P} \langle x, v \rangle - \min_{x \in P} \langle x, v \rangle$ . Here  $\langle x, v \rangle$  is the standard inner product in  $\mathbb{R}^d$ . The smallest lattice width over all primitive  $v \in \mathbb{Z}^d$  is called the *lattice width of  $P$*  and is denoted  $w(P)$ .

**Proposition 3.1.** *Let  $P \subset \mathbb{R}^2$  be a lattice polygon. Then  $\lambda(P) \leq \frac{2\text{Vol}_2(P)}{w(P)}$  where  $\text{Vol}_2(P)$  is the Euclidean area and  $w(P)$  the lattice width of  $P$ .*

*Proof.* By the proof of Theorem 2.15,  $\lambda(P) = |Z|$  for some rational zonotope  $Z \subseteq P$  with at most 3 distinct summands, i.e.

$$Z = a + \alpha_1[0, v_1] + \alpha_2[0, v_2] + \alpha_3[0, v_3],$$

where  $v_i \in \mathbb{Z}^2$  are distinct primitive vectors,  $a \in \mathbb{Q}^2$ , and  $\alpha_i \in \mathbb{Q}$ . We have  $|Z| = \alpha_1 + \alpha_2 + \alpha_3$ .

Let  $w_i$  be the lattice width of  $P$  in the direction of a primitive vector  $v_i^\perp$ , orthogonal to  $v_i$ . We claim that

$$(3.1) \quad \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 \leq 2 \text{Vol}_2(P).$$

Indeed, let  $A_i$  (resp.  $B_i$ ) be a vertex of  $P$  where the inner product  $\langle x, v_i^\perp \rangle$  attains its minimum (resp. maximum). Similarly, let  $E_i$  (resp.  $I_i$ ) be the side of  $Z$  where  $\langle x, v_i^\perp \rangle$  attains its minimum (resp. maximum). Connect  $A_i$  to  $E_i$  and  $B_i$  to  $I_i$  for  $i = 1, 2, 3$  by line segments. Also triangulate  $Z$  (if it is not one-dimensional) by drawing the diagonals through the center of  $Z$ . We obtain a (not necessarily convex) triangulated polygon  $S$  inside  $P$ , see Figure 1.

Note that the sum of the areas of the four triangles with bases  $E_i$  and  $I_i$  equals  $\frac{1}{2} \alpha_i w_i$ . Therefore, the left hand side of (3.1) represents twice the area of  $S$ , and (3.1) follows.

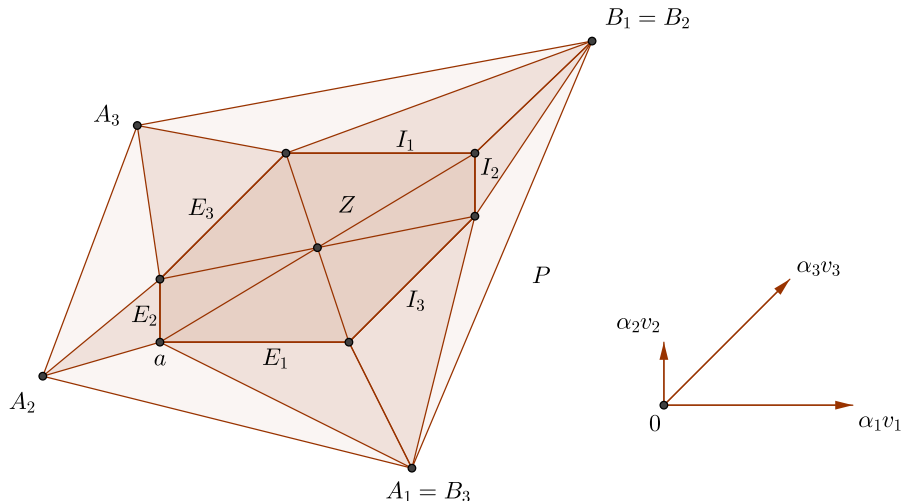
It remains to note that  $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 \geq (\alpha_1 + \alpha_2 + \alpha_3) w(P) = \lambda(P) w(P)$ .  $\square$

Below we apply this bound to give an explicit formula for the (rational) Minkowski length of any triangle. Recall the *lattice diameter*  $\ell(P)$  and the *rational diameter*  $s(P)$  defined in Section 2.1.

**Corollary 3.2.** *Let  $T \subset \mathbb{R}^2$  be a lattice triangle. Let  $s(T)$  be its rational diameter and  $\ell(T)$  its lattice diameter. Then*

$$\lambda(T) = s(T) \quad \text{and} \quad L(T) = \lfloor s(T) \rfloor = \ell(T).$$

*Consequently,  $\lambda(tT) = s(T)t$  and  $L(tT) = \lfloor s(T)t \rfloor$ .*

FIGURE 1. A triangulated polygon inside  $P$ .

*Proof.* Let  $v \in \mathbb{Z}^2$  be a primitive vector such that the lattice width of  $T$  in the direction orthogonal to  $v$  equals  $w(T)$ . Then  $s_v(T)w(T) = 2 \text{Vol}_2(T)$ , where  $s_v(T)$  as in Definition 2.10. It follows that  $s_v(T)$  is, in fact,  $s(T)$ . Applying Proposition 3.1, we get  $\lambda(T) \leq s(T)$ . Conversely,  $T$  contains the segment  $E$  parallel to  $[0, s(T)v]$ . Therefore, by the proof of Theorem 2.15,  $\lambda(T) \geq |E| = s(T)$ .

As for  $L(T)$ , it is clear that  $L(T) \leq \lambda(T) = s(T)$ , by the definition of  $\lambda(P)$ . Thus,  $L(T) \leq \lfloor s(T) \rfloor$ . On the other hand,  $T$  contains a translation of the lattice segment  $[0, \lfloor s(T) \rfloor v]$ , hence  $L(T) \geq \lfloor s(T) \rfloor$ .

Finally, by above,  $\lambda(tT) = s(tT) = ts(T)$  and  $L(tT) = \lfloor s(tT) \rfloor = \lfloor ts(T) \rfloor$ .  $\square$

**Remark 3.3.** The above proof shows that our bound in Proposition 3.1 is tight, as  $\lambda(T) = s(T) = 2 \text{Vol}_2(T)/w(T)$  for any lattice triangle  $T$ .

#### 4. EXAMPLES AND OPEN PROBLEMS

In this section we illustrate our results with several examples and raise some questions.

Our first example shows that  $L(tP)$  can have an arbitrarily large period  $k$ .

**Example 4.1.** Let  $T_k \subset \mathbb{R}^2$  denote the triangle with vertices  $(0,0)$ ,  $(k,1)$ , and  $(1,k)$ , for  $k \geq 2$ . It is not hard to see that  $\ell(T_k) = k - 1$  and  $s(T_k) = k - 1/k$ . By Corollary 3.2,

$$L(tT_k) = \left\lfloor \left(k - \frac{1}{k}\right)t \right\rfloor,$$

which is a quasi-linear function with period  $k$ .

**Example 4.2.** Let  $P$  be a square with vertices  $(2,0)$ ,  $(3,2)$ ,  $(1,3)$ , and  $(0,1)$ . One readily sees that  $\text{Vol}_2(P) = 5$ ,  $w(P) = 3$ , and  $L(P) = 3$ . Therefore, by Proposition 3.1,  $\lambda(P) \leq 10/3$ .

Note that  $3P$  contains a zonotope  $Z$  with  $|Z| = 10$  (in fact,  $Z$  is a square with vertices  $(2, 2)$ ,  $(7, 2)$ ,  $(2, 7)$ , and  $(7, 7)$ ). Therefore,  $10 \leq L(3P) \leq 3\lambda(P) \leq 10$ , which implies that  $\lambda(P) = 10/3$  and  $L(tP)$  has period  $k = 3$ . By Lemma 2.18, if  $t = 3q$  then  $L(tP) = 10q$ . Now if  $t = 3q + 1$  we have

$$10q + 3 = L(3qP) + L(P) \leq L(tP) \leq 10t/3 = 10q + 10/3,$$

and, hence,  $L(tP) = 10q + 3$ . Similarly,  $L(tP) = 10q + 6$  when  $t = 3q + 2$ . Therefore,

$$L(tP) = 10 \left\lfloor \frac{t}{3} \right\rfloor + 3r, \quad \text{if } t \equiv r \pmod{3}.$$

Looking at the above examples one may suspect that  $L(P) = \lfloor \lambda(P) \rfloor$  for any polytope  $P$ . This would imply that  $L(tP)$  is not just eventually quasi-linear, but quasi-linear:

$$L(tP) = k\lambda(P) \left\lfloor \frac{t}{k} \right\rfloor + L(rP),$$

for any  $t \equiv r \pmod{k}$  (see Theorem 2.20).

However,  $L(P) = \lfloor \lambda(P) \rfloor$  does not hold even for the case of lattice polygons, as demonstrated by the following example.

**Example 4.3.** Let  $P = 2Q$  where  $Q$  is the square with vertices  $(1, 0)$ ,  $(5, 1)$ ,  $(4, 5)$ , and  $(0, 4)$ . Then by Proposition 3.1,

$$\lambda(P) = 2\lambda(Q) \leq \frac{68}{5}.$$

Also  $L(5P)$  contains  $Z$  with  $|Z| = 68$  (namely,  $Z$  is the square with vertices  $(8, 8)$ ,  $(8, 42)$ ,  $(42, 42)$ , and  $(42, 8)$ ), hence,  $\lambda(P) = 68/5$ . By observation,  $L(P) = 12$ , which illustrates that  $\lambda(P) - L(P)$  can be as large as  $8/5$ .

**Problem 1.** Find the supremum of  $\lambda(P) - L(P)$  over all lattice polytopes  $P \subset \mathbb{R}^d$ .

It is not hard to see that  $\lambda(P) - L(P) < 4$  for any lattice polygon  $P \subset \mathbb{R}^2$ , but we are confident that this bound could be improved.

In all 2-dimensional examples we computed, the function  $L(tP)$  was always quasi-linear. Although we do not expect this to be the case in general, we have not been able to produce a counterexample.

**Problem 2.** Prove that  $L(tP)$  is quasi-linear or give an example of a lattice polytope  $P$  for which  $L(tP)$  is not quasi-linear.

Finally, we have seen that  $L(P) = L_1(P)$  when  $P$  is a positive integer dilate of a unimodular simplex as well as when  $P$  is any simplex in dimension two. This prompts the following problem.

**Problem 3.** Prove or disprove that for any simplex in  $\mathbb{R}^d$  its Minkowski length coincides with its lattice diameter.

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